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Quantum N-colour Ashkin–Teller model: exact solution in the large-N limit

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Abstract. The statistical mechanics of the quantum N-colour Ashkin–Teller model is solved exactly in the large-N limit. It is shown that the results can be expressed in terms of the partition function of some effective Ising model in a transverse field. The structure of the ground state is found to be equivalent to the phase diagram of the classical system

The classical Ashkin–Teller model [1] (ATM) has many interesting properties—self-duality, line of continuously varying exponents, bifurcation points in the phase diagram, etc—which has justified its intensitive study over the past years [2–4]. This model can be viewed [5] as two Ising models coupled through their energy densities, and it reduces to the Ising and the four-state Potts models for particular values of the couplings. Generalizations of the ATM containing $N > 2$ interacting Ising systems (N-colour ATM) have also been considered in the literature [6], and the solution for large N has been obtained [7].

The Hamiltonian description of the ATM—obtained by taking the time-continuum limit [8]—maps the original two-dimensional classical system onto a one-dimensional quantum problem [9]. This quantum ATM shows an even richer behaviour than its classical counterpart. For ferromagnetic couplings the ground-state phase structure is similar to the phase diagram of the classical ATM, while for antiferromagnetic couplings it shows a critical surface, whose borders are lines of Kosterlitz–Thouless transition points and a line of first-order transition points [9, 10]. In a restricted parameter space, where one loses part of the relevant features of the model, the continuum version of the quantum ATM is the massive Thirring model. This connection has been exploited to obtain some of the continuously varying critical exponents as a function of the four-spin coupling [11]. For $N > 2$ the continuum limit is the Gross–Neveu model, a relationship which has provided further results [12] from the study of this field-theoretical model.

In this work we are going to study the *quantum* N-colour ATM in the large-N limit. We will see that the statistical mechanics of this system can be solved exactly and we will investigate its ground-state properties, which are expected to be equivalent to those of the free energy of the classical large-N ATM [7]. We will be mainly interested in answering the question of whether the existence of infinitely many spin operators at every site would produce a sort of mean-field behaviour, with a phase transition at

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finite temperature for this one-dimensional quantum system. On the other hand, we want to investigate whether the richer phase structure of the Hamiltonian description for antiferromagnetic couplings survives the limit $N \to \infty$. In principle this model affords a comparison between the (exactly solvable) two-dimensional-classical and one-dimensional-quantum versions of a non-trivial system. Unfortunately, we will see that the large-N limit is strong enough to wash out subtle differences and only the known equivalence of the quantum and classical descriptions of some effective Ising model remains.

The quantum N-colour Ashkin–Teller model is defined by the Hamiltonian

$$H = H^z + H^x$$
$$= -\sum_{i=1}^{M}\left[K s_i^z \cdot s_{i+1}^z + \frac{g}{2N}(s_i^z \cdot s_{i+1}^z)^2\right] - \sum_{i=1}^{M}\left[h e \cdot s_i^x + \frac{\tilde{g}}{2N}(e \cdot s_i^x)^2\right]$$

where $s^a = (\sigma_1^a, \sigma_2^a, \ldots, \sigma_N^a)$, $e = (1, 1, \ldots, 1)$ and the σ^a $(a = x, z)$ are the Pauli matrices. The scalar product is the usual one in colour space.

By using the Trotter formula [13] the partition function corresponding to H can be written

$$Z = \mathrm{Tr}\exp(-\beta H) = \lim_{n\to\infty} Z^{(n)} \qquad (\beta = 1/kT)$$

where $Z^{(n)}$ is the n-approximant

$$Z^{(n)} = \mathrm{Tr}\left[\exp\left(-\frac{\beta}{n}H^z\right)\exp\left(-\frac{\beta}{n}H^x\right)\right]^n \tag{1}$$

Furthermore, by inserting n complete sets of eigenstates of s^z,

$$(s_i^z|\alpha_j\rangle = s_{ij}|\alpha_j\rangle \; i = 1, M; \; j = 1, n)$$

we obtain

$$Z^{(n)} = \sum_{\{\alpha_j\}}\prod_{j=1}^{n}\exp\left[-\frac{\beta}{n}H^z(\alpha_j)\right]\left\langle\alpha_j\left|\exp\left(-\frac{\beta}{n}H^x\right)\right|\alpha_{j+1}\right\rangle. \tag{2}$$

As usual, $\{\alpha_j\}$ denotes all possible spin configurations α_j and periodic boundary conditions in the Trotter direction $(\alpha_{n+1} = \alpha_1)$ take care of the trace in (1).

To evaluate the off-diagonal terms in (2) we perform a Hubbard–Stratonovich transformation of H^x, so that

$$\prod_{j=1}^{n}\left\langle\alpha_j\left|\exp\left(-\frac{\beta}{n}H^x\right)\right|\alpha_{j+1}\right\rangle$$
$$= \prod_{j=1}^{n}\prod_{i=1}^{M}\int\frac{\mathrm{d}\tilde{\varphi}_{ij}}{(2\pi)^{1/2}}\exp\left[-\frac{\beta N}{2n}\tilde{\varphi}_{ij}^2 + \frac{1}{2}\log\left(\frac{N\beta}{n}\right)\right]$$
$$\times\left\langle s_{ij}\left|\exp\left[\frac{\beta}{n}(h + \tilde{g}^{1/2}\tilde{\varphi}_{ij})e \cdot s_i^x\right]\right|s_{i,j+1}\right\rangle$$
$$= \int[\mathrm{d}\tilde{\varphi}]\exp\left[-S(\tilde{\varphi}, s) + \frac{Mn}{2}\log\left(\frac{N\beta}{n}\right)\right]$$

where $[d\tilde{\varphi}] = \Pi_{j=1}^{n} \Pi_{i=1}^{M} d\tilde{\varphi}_{ij}/(2\pi)^{1/2}$ and

$$S(\tilde{\varphi}, s) = \sum_{i,j} \left\{ \frac{N\beta}{2n} \tilde{\varphi}_{ij}^2 - \frac{N}{2} \log \tfrac{1}{2} \sinh\left[\frac{2\beta}{n}(h + \tilde{g}^{1/2}\tilde{\varphi}_{ij})\right] - \tfrac{1}{2}\log \coth\left[\frac{\beta}{n}(h + \tilde{g}^{1/2}\tilde{\varphi}_{ij})\right] s_{ij} \cdot s_{i,j+1} \right\}.$$

Performing a similar decoupling of the four-spin term in H^z, (2) reduces finally to

$$Z^{(n)} = \int [d\varphi\, d\tilde{\varphi}] \exp\left[-NS(\varphi, \tilde{\varphi}) + Mn \log\left(\frac{N\beta}{n}\right)\right]$$

with

$$S(\varphi, \tilde{\varphi}) = \frac{\beta}{2n} \sum_{i,j} (\varphi_{ij}^2 + \tilde{\varphi}_{ij}^2) - \log Z_{\mathrm{QI}}^{(n)}[\beta(K + g^{1/2}\varphi_{ij}), \beta(h + \tilde{g}^{1/2}\tilde{\varphi}_{ij})].$$

In the last equation $Z_{\mathrm{QI}}^{(n)}$ is the n-approximant to the partition function of the quantum Ising model in a transverse field (QIMTF) [13], evaluated for some effective inhomogeneous spin-coupling and field.

Up to this point the number of colours N is a free parameter without any particular role in the calculations. However, if we now take N going to infinity the previous expression for $Z^{(n)}$ can be evaluated exactly by performing a saddle-point calculation†. Thus, we obtain

$$f = \lim_{M\to\infty} \lim_{n\to\infty} \lim_{N\to\infty} \left(-\frac{1}{MN\beta}\right) \log Z^{(n)}$$
$$= \tfrac{1}{2}(\varphi^2 + \tilde{\varphi}^2) + f_{\mathrm{QI}}(\beta J, \beta\Gamma) \tag{3}$$

where $f_{\mathrm{QI}}(\beta J, \beta\Gamma)$ is the free energy per spin of the QIMTF with the effective coupling J and transverse field Γ given by

$$J = K + g^{1/2}\varphi \qquad \Gamma = h + \tilde{g}^{1/2}\tilde{\varphi}.$$

From [14] we know that

$$f_{\mathrm{QI}}(\beta J, \beta\Gamma) = -\frac{1}{2\pi\beta} \int_0^{2\pi} \log[2\cosh \beta\omega(q)]\, dq \tag{4}$$

where

$$\omega(q) = (J^2 + \Gamma^2 + 2J\Gamma \cos q)^{1/2}.$$

The values φ and $\tilde{\varphi}$ of the frozen fields are obtained from the saddle-point equations $\partial S/\partial\varphi = \partial S/\partial\tilde{\varphi} = 0$. In terms of J and Γ these equations read

$$J = K + \frac{g}{N} \langle s_i^z \cdot s_{i+1}^z \rangle_{\mathrm{QI}}$$
$$= K + \frac{g}{2\pi} \int_0^{2\pi} \tanh[\beta\omega(q)] \frac{(J + \Gamma \cos q)}{\omega(q)}\, dq \tag{5a}$$

† The interchange of the Trotter and large-N limits is safe because of the simple form N enters the calculation. A justification of this interchange is provided by the fact that the contribution from Gaussian fluctuations (which is of order $1/N$) does not diverge for $n \to \infty$.

and

$$\Gamma = h + \frac{\tilde{g}}{N}\langle e \cdot s_i^x\rangle_{QI}$$
$$= h + \frac{\tilde{g}}{2\pi}\int_0^{2\pi} \tanh[\beta\omega(q)]\,\frac{(\Gamma + J\cos q)}{\omega(q)}\,dq. \tag{5b}$$

From the fact that $f_{QI}(\beta J, \beta\Gamma)$ is an analytic function of β for all $\beta < \infty$ we conclude that the free energy (3), together with the definitions (5) of J and Γ, shows no singular behaviour and, hence, there is no phase transition at finite temperature.

We turn now to the investigation of the phase structure of the ground state. In the following we will measure the energy in units of the external field ($h = 1$) and, for simplicity, we will restrict ourselves to the plane $\tilde{g} = g$ in parameter space. This is not an important restriction since this plane already contains the main features of the phase diagram even for finite N [9, 10].

Defining $\lambda = J/\Gamma$, for $\beta \to \infty$ the system of equations (3)–(5) becomes

$$\varepsilon_0 = \lim_{\beta\to\infty} f = \frac{1}{2g}(J-K)^2 + \frac{1}{2g\lambda^2}(J-\lambda)^2 - \frac{2J}{\pi}\frac{(1+\lambda)}{\lambda}E(\gamma) \tag{6a}$$

$$\lambda = K + \frac{2g}{\pi}\frac{(1-\lambda)}{2\lambda}[(1+\lambda)^2E(\gamma) - (1+\lambda^2)F(\gamma)] \tag{6b}$$

$$J = K + \frac{2g}{\pi}\frac{1}{2\lambda}[(1+\lambda)E(\gamma) - (1-\lambda)F(\gamma)] \tag{6c}$$

where $F(\gamma)$ and $E(\gamma)$ are the complete elliptic integrals of the first and second kind respectively, and the modulus $\gamma^2 = 4\lambda/(1+\lambda)^2$.

The phase structure of the ground state can now be obtained by numerically studying equations (6). However, we know [15] that the eliptic integrals have a non-analytic behaviour for $\gamma \to 1$, which corresponds to the critical point $\lambda_c = 1$ of the effective QIMTF in (3). So, we will see that the phase diagram can be characterized to a large extent by analytically studying these equations for $\lambda \sim 1$. Our discussion will parallel that of [7].

By setting $\lambda = 1 - \delta$ ($\delta \sim 0$), (6b) can be expanded as

$$1 - K = \left[1 + \frac{2g}{\pi}(2 - 3\log 2) + \frac{2g}{\pi}\log|\delta|\right]\delta + O(\delta^2). \tag{7}$$

For $g < 0$ the only solution of this equation consistent with the condition $\delta \sim 0$ is

$$\delta \simeq \frac{\pi}{2g}\Delta K \frac{1}{(\log|\pi/2g\,\Delta K| + \pi/2g + 2 - 3\log 2)} \tag{8}$$

with $\Delta K = (1-K) \sim 0$. From (8) we see that the Ising critical condition $\lambda_c = 1$ ($\delta = 0$) corresponds to $K_c = 1$, so that the line $g < 0$, $K_c = 1$ is a line of second-order phase transitions.

The point $g = 0$ is a decoupling point where the system behaves like an infinite collection of independent Ising systems. To understand what happens for $g > 0$ it is convenient to consider a small region $g \sim 0^+$ fixed, $K \to 1$. In this case, in addition to (8) there are other two solutions to (7) consistent with the condition $\delta \sim 0$, namely

$$\delta \simeq \pm 8\exp\left[-2\left(1 + \frac{\pi}{4g}\right)\right] + \frac{\pi}{2g}\Delta K. \tag{9}$$

To find out which one corresponds to the absolute minimum of the ground-state energy we expand (6*a*) in powers of δ and ΔK:

$$\varepsilon_0 = -\frac{4}{\pi}\left(1+\frac{g}{\pi}\right)+\frac{2}{\pi}\Delta K-\frac{1}{g}\left(1+\frac{g}{\pi}\log|\delta|\right)\delta\Delta K+\frac{3}{2\pi}\log|\delta|\delta^2$$

$$+\frac{g}{\pi^2}\log^2|\delta|\delta^2+O(\Delta^2 K, \delta\Delta K, \delta^2).$$

From this equation one can see that (8) is no longer the absolute minimum, which has now shifted to (9) with the plus sign for $\Delta K>0$ and with the minus sign for $\Delta K<0$. Again, in the small region $g\sim 0^+$ the line $K=1$ is a phase boundary, where the energies corresponding to both solutions (9) merge to a single value. However, in this case $\delta\neq 0$ on the phase transition line, so that $\lambda<1$ in one of the phases and $\lambda>1$ in the other. The effective QIMTF in (3) jumps from a disordered to a partially ordered phase, implying a first-order transition. For g large enough, $\delta\gg 1$ and (7) breaks down. The first-order transition line departs from the line $K=1$ and can be found numerically by equating the energies of the continuation of solutions (9) for large g. This picture has been confirmed numerically.

Several quantities of interest can now be obtained from equations (3–6). For instance, the ground-state correlation function of the z-component of the spin. For $\lambda>1$ and $|i-j|\to\infty$ it behaves like

$$\langle 0|s_i^z\cdot s_j^z|0\rangle=\langle 0|s_i^z\cdot s_j^z|0\rangle_{\mathrm{QI}}=N(1-\lambda^{-2})^{1/4}.$$

From this equation we obtain for the magnetization in the z-direction the Ising-like result $M_z=(1-\lambda^{-2})^{1/8}$. According to (9) the jump in M_z at the first-order transition line is given by

$$\Delta M_z\simeq\sqrt{2}\exp\left[-\frac{1}{4}\left(+\frac{\pi}{4g}\right)\right] \qquad (K=1, g\sim 0^+)$$

a quantity which is similar to the latent heat in the classical system. For the magnetization in the transverse direction we obtain

$$M_x=-\frac{\partial\varepsilon_0}{\partial h}\simeq\frac{2}{\pi}+\frac{\pi}{4g^2}\Delta K \qquad (K\sim 1, g\sim 0).$$

It is also interesting to mention that the second derivative of the free energy (3) with respect to K does not diverge at the second-order transition line, which is equivalent to the fact that the specific heat of the classical system is finite at the corresponding phase transition line [7]. This is related to the presence of $\log\Delta K$ in the denominator of (8), which cancels the logarithmic divergence of the specific heat of the effective Ising model.

In conclusion, we have solved the statistical mechanics of the quantum N-colour Ashkin–Teller model in the limit $N\to\infty$. We have shown that the model does not display any phase transition at finite temperature, despite the fact that it involves the interaction of infinitely many spin operators at every site. We also found that the ground state has a structure entirely analogous to the phase diagram of the classical system [7], with the rich behaviour found for antiferromagnetic couplings [9] being washed out by the large n-limit.

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